Eureka Math™ Homework Helper

2015–2016

Algebra I
Module 3
Lessons 1–7

Eureka Math, A Story of Functions®

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Lesson 1: Integer Sequences—Should You Believe in Patterns?

Generating Terms of a Sequence

1. Consider a sequence given by the formula $f(n) = 12 - 7n$ starting with $n = 1$. Generate the first 5 terms of the sequence.

$\boldsymbol{f(1) = 12 - 7(1) = 5}$

$\boldsymbol{f(2) = 12 - 7(2) = -2}$

$\boldsymbol{f(3) = 12 - 7(3) = -9}$

$\boldsymbol{f(4) = 12 - 7(4) = -16}$

$\boldsymbol{f(5) = 12 - 7(5) = -23}$

I see that this sequence has a "minus 7" pattern. I could use this pattern to continue generating terms in the sequence.

To find the first five terms of the sequence, I can replace n with the numbers 1, 2, 3, 4, and 5.

The first five terms of the sequence are $\boldsymbol{5, -2, -9, -16, -23}$.

Writing a Formula for a Sequence

2. Consider the following sequence that follows a times $\frac{1}{2}$ pattern: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$.

   a. Write a formula for the n^{th} term of the sequence. Be sure to specify what value of n your formula starts with.

$\boldsymbol{f(n) = \left(\frac{1}{2}\right)^{n-1}}$ ***starting with*** $\boldsymbol{n = 1}$

I know that my formula can start with any value of n but that the convention is to start with $n = 1$.

I can check my formula by using it to generate the first few terms of the sequence.

$$f(1) = \left(\frac{1}{2}\right)^{1-1} = \left(\frac{1}{2}\right)^{0} = 1$$

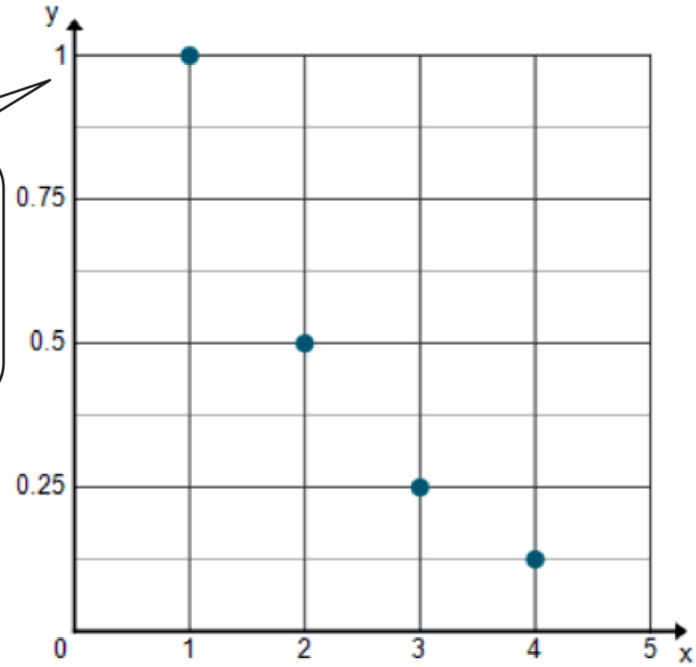
$$f(2) = \left(\frac{1}{2}\right)^{2-1} = \left(\frac{1}{2}\right)^{1} = \frac{1}{2}$$

b. Using the formula, find the 10^{th} term of the sequence.

$$\boldsymbol{f(10)} = \left(\frac{1}{2}\right)^{10-1} = \left(\frac{1}{2}\right)^{9} = \frac{1}{512}$$

Since my formula starts with $n = 1$, I can find the 10^{th} term by replacing n with 10.

c. Graph the four terms of the sequence as ordered pairs $(n, f(n))$ on a coordinate plane.



The ordered pair for the first term is $(1, 1)$. The ordered pair for the second term is $\left(2, \frac{1}{2}\right)$, and so on.

Lesson 2: Recursive Formulas for Sequences

Generating Terms of a Sequence When Given a Recursive Formula

List the first five terms of each sequence.

1. $f(n) = -3f(n-1)$ and $f(1) = 2$ for $n \geq 2$

I can find the next term in the sequence by starting with $n = 2$.

I see that this is a recursive formula. To find any term in the sequence, I need to use the previous term.

This means that the first term of the sequence is 2. I can use that to get started.

I replaced n with 2 in the formula to find the second term.

$\boldsymbol{f(1) = 2}$

$\boldsymbol{f(2) = -3f(2-1) = -3f(1) = -3(2) = -6}$

$\boldsymbol{f(3) = -3f(3-1) = -3f(2) = -3(-6) = 18}$

$\boldsymbol{f(4) = -3f(4-1) = -3f(3) = -3(18) = -54}$

$\boldsymbol{f(5) = -3f(5-1) = -3f(4) = -3(-54) = 162}$

The first five terms of the sequence are $\boldsymbol{2, -6, 18, -54, 162.}$

2. $a_{n+1} = a_n + 2n + 1$ where $a_1 = 1$ for $n \geq 1$

This sequence is valid for integers greater than or equal to 1.

The subscripts in this notation represent the term number just like the values in the parentheses did in the formula above.

I replaced n with 1 in the formula to find the second term.

$\boldsymbol{a_1 = 1}$

$\boldsymbol{a_2 = a_1 + 2(1) + 1 = 1 + 2 + 1 = 4}$

$\boldsymbol{a_3 = a_2 + 2(2) + 1 = 4 + 4 + 1 = 9}$

$\boldsymbol{a_4 = a_3 + 2(3) + 1 = 9 + 6 + 1 = 16}$

$\boldsymbol{a_5 = a_4 + 2(4) + 1 = 16 + 8 + 1 = 25}$

The first five terms of the sequence are $\boldsymbol{1, 4, 9, 16, 25.}$

Writing a Recursive Formula for a Sequence

3. Write a recursive formula for the sequence that has an explicit formula $f(n) = 4n - 2$ for $n \geq 1$.

$\boldsymbol{f(1) = 4(1) - 2 = 2}$

$\boldsymbol{f(2) = 4(2) - 2 = 6}$

$\boldsymbol{f(3) = 4(3) - 2 = 10}$

$\boldsymbol{f(4) = 4(4) - 2 = 14}$

I see that this sequence is following a "plus 4" pattern.

It might be helpful to generate the first few terms of the sequence.

$\boldsymbol{f(n+1) = f(n) + 4}$ ***where*** $\boldsymbol{f(1) = 2}$ ***and*** $\boldsymbol{n \geq 1}$

4. The bacteria culture has an initial population of 100 and it quadruples in size every hour.

This sequence has a "times $\boldsymbol{4}$***" pattern:*** $\boldsymbol{100, 400, 1600, 6400}$

$\boldsymbol{B_{n+1} = 4B_n}$ ***where*** $\boldsymbol{B_1 = 100}$ ***and*** $\boldsymbol{n \geq 1}$

I can use subscripts or parentheses like $B(n+1)$ to name the sequence.

Each term in the sequence is 4 times the previous one.

$$400 = 4 \cdot 100$$
$$1600 = 4 \cdot 400$$
$$6400 = 4 \cdot 1600$$

Noticing this pattern helps me write the recursive formula.

Lesson 3: Arithmetic and Geometric Sequences

Identifying Sequences as Arithmetic or Geometric

1. List the first five terms of the sequence given below, and identify it as arithmetic or geometric.

$$A(n+1) = -3 \cdot A(n) \text{ for } n \geq 1 \text{ and } A(1) = 2$$

$\mathbf{A(1) = 2}$

$\mathbf{A(2) = -3 \cdot A(1) = -3 \cdot 2 = -6}$

$\mathbf{A(3) = -3 \cdot A(2) = -3 \cdot -6 = 18}$

$\mathbf{A(4) = -3 \cdot A(3) = -3 \cdot 18 = -54}$

$\mathbf{A(5) = -3 \cdot A(4) = -3 \cdot -54 = 162}$

I was given a recursive formula and the first term, $A(1)$. I can use the first term to find the second term.

This sequence is geometric.

I see that each term in the sequence is the product of the previous term and -3.

2. Identify the sequence as arithmetic or geometric, and write a recursive formula for the sequence. Be sure to identify your starting value.

$$15, 9, 3, -3, -9, \ldots$$

This sequence is arithmetic.

I see that each term in the sequence is the sum of the previous term and -6.

$\mathbf{f(n+1) = f(n) - 6 \textit{ for } n \geq 1 \textit{ and } f(1) = 15}$

Writing the Explicit Form of an Arithmetic or Geometric Sequence

3. Consider the arithmetic sequence $15, 9, 3, -3, -9, \ldots$.
   a. Find an explicit form for the sequence in terms of n.

   $\mathbf{f(n) = 15 + (n-1) \cdot -6 = -6n + 21 \textit{ for } n \geq 1}$

I need to identify the pattern. To find the second term, I need to subtract 6 one time. To find the third term, I need to subtract 6 two times. To find the n^{th} term, I need to subtract 6 $(n-1)$ times.

b. Find the 80^{th} term.

$$\boldsymbol{f(80) = -6(80) + 21 = -459}$$

To find the 80^{th} term, I need to find $f(80)$ using the explicit form.

4. Find the common ratio and an explicit form for the following geometric sequence.

$108, 36, 12, 4, \ldots$

I can find the common ratio, r, by dividing any two successive terms.

$$\boldsymbol{r = \frac{36}{108} = \frac{12}{36} = \frac{4}{12} = \frac{1}{3}}$$

I need to identify the pattern. To find the second term, I need to multiply by $\frac{1}{3}$ one time. To find the third term, I need to multiply by $\frac{1}{3}$ two times. To find the n^{th} term, I need to multiply by $\frac{1}{3}$ $(n-1)$

$$\boldsymbol{f(n) = 108\left(\frac{1}{3}\right)^{n-1}}$$

I can check my formula by finding a term in the sequence.

$f(4) = 108\left(\frac{1}{3}\right)^{4-1} = 108\left(\frac{1}{3}\right)^{3} = 4$

The fourth term of the sequence is 4.

5. The first term in an arithmetic sequence is 2, and the 5^{th} term is 8. Find an explicit form for the arithmetic sequence.

$\boldsymbol{8 = 2 + 4 \cdot d}$

$\boldsymbol{\frac{3}{2} = d}$

$2, \_, \_, \_, 8$

I need to find some number, d, that when added to the first term 4 times results in the fifth term.

$$\boldsymbol{f(n) = 2 + (n-1)\cdot\frac{3}{2} = \frac{3}{2}n + \frac{1}{2}}$$

I can check my formula by finding a term in the sequence.

$f(5) = \frac{3}{2}(5) + \frac{1}{2} = 8$

The fifth term of the sequence is 8.

Lesson 4: Why Do Banks Pay YOU to Provide Their Services?

Calculations Involving Simple Interest

1. $800 is invested at a bank that pays 5% simple interest. Calculate the amount of money in the account after 12 years.

I know that simple interest means that interest is earned only on the original investment amount.

$\boldsymbol{I(t) = Prt}$

I can use this formula to calculate the interest. P is the principal amount, and r is the interest rate in decimal form.

$\boldsymbol{I(12) = 800(0.05)(12)}$

$\boldsymbol{I(12) = 480}$

After 12 years, the account will have $\$1,280$.

Calculations Involving Compound Interest

2. $800 is invested at a bank that pays 5% interest compounded annually. Calculate the amount of money in the account after 12 years.

I know that compound interest means that each time interest is earned, it becomes part of the principal.

$\boldsymbol{FV = P(1+r)^n}$

I can use this formula to calculate the future value n years after investing.

$\boldsymbol{FV = 800(1+0.05)^{12}}$

$\boldsymbol{\approx 1,436.69}$

After 12 years, the account will have $\$1,436.69$.

Lesson 5: The Power of Exponential Growth

1. In the year 2000, a total of 768,586 high school students took an Advanced Placement (AP) exam. Since the year 2000, the number of high school students who take an AP exam has increased at an approximate rate of 9% per year.

   a. What explicit formula models this situation?

   $f(t) = 768586(1.09)^t$,

   where t represents the number of years since 2000.

   This is an example of exponential growth, so I need an explicit formula for a geometric sequence to model this situation.

   In this formula, I am starting with $t = 0$ (the year 2000).

   b. If this trend continues, predict the number of high school students who will take an AP exam in the year 2020.

   $f(20) = 768586(1.09)^{20} = 4307471.654$

   Since t represents years since 2000, I need to evaluate $f(20)$.

   If this trend continues, approximately 4,307,472 students will take an AP exam in the year 2020.

2. Jackie decided to start a savings plan where she deposited \$0.01 in a jar on day one, \$0.03 on day two, \$0.09 on day three, and so on, tripling the amount she saved each day. After how many days of following this plan would the amount she deposited in the jar exceed \$10,000? Be sure to include an explicit formula with your response.

   $A(n) = 0.01(3)^{n-1}$ ***for*** $n \geq 1$

   I see that the amount she saves each day forms a geometric sequence with a "times 3" pattern.

   I can check my formula.
   Day 1: $A(1) = 0.01(3)^{1-1} = 0.01$
   Day 2: $A(2) = 0.01(3)^{2-1} = 0.03$
   Day 3: $A(3) = 0.01(3)^{3-1} = 0.09$

   $A(13) = 0.01(3)^{13-1} = 0.01(3)^{12} = 5314.41$

   $A(14) = 0.01(3)^{14-1} = 0.01(3)^{13} = 15943.23$

   I used trial and error to find the answer.

   On day 14 of the savings plan, the amount she deposited would exceed \$10,000.

Lesson 6: Exponential Growth—U.S. Population and World Population

Stuart plans to deposit $\$1{,}000$ into a savings account. His bank offers two different types of savings accounts.

Option A pays a simple interest rate of 3.2% per year. Option B pays a compound interest rate of 2.8% per year, compounded monthly.

a. Write an explicit formula for the sequence that models the balance in Stuart's account t years after he deposits the money if he chooses option A.

$\boldsymbol{A(t) = 1000 + 1000(0.032)t}$

I know that simple interest means that the same amount of interest will be added each year. I can use the formula $I = Prt$ to write an expression for the total interest.

b. Write an explicit formula for the sequence that models the balance in Stuart's account m months after he deposits the money if he chooses option B.

$\boldsymbol{B(t) = 1000\left(1 + \frac{0.028}{12}\right)^{m}}$

Since the interest is compounded monthly, I need to divide the annual interest rate by 12 to find the interest rate per month.

c. Which option is represented with a linear model? Why?

Option A is represented with a linear model because there is a constant rate of change each year.

d. Which option is represented with an exponential model? Why?

Option B is represented with an exponential model because there is a constant ratio of change each month.

I need to determine when $A(t)$ and $B(t)$ will equal \$2000.

e. Approximately how long will it take Stuart to double his money if he chooses option A? Option B?

$$A(t) = 1000 + 1000(0.032)t$$
$$2000 = 1000 + 1000(0.032)t$$
$$t = 31.25$$

I can solve this equation for t.

$$B(t) = 1000\left(1 + \frac{0.028}{12}\right)^m$$
$$2000 = 1000\left(1 + \frac{0.028}{12}\right)^m$$
$$m \approx 297$$

I need to use trial and error to find m.

If he chooses option A, it will take him 32 years to double his money. If he chooses option B, it will take him 24 years and 9 months to double his money.

f. How should Stuart decide between the two options?

If he is going to invest for a short amount of time (fewer than 10 years), he should choose option A. If he is going to invest for a long amount of time (10 years or longer), he should choose option B.

$$A(10) = 1000 + 1000(0.032)10 = 1320$$
$$B(120) = 1000\left(1 + \frac{0.028}{12}\right)^{120} \approx 1322.70$$

At 10 years (120 months), the balance in the account for option B is larger than the balance in the account for option A.

Lesson 7: Exponential Decay

1. Since 1950, the population of Detroit has been decreasing. The population of Detroit (in millions) can be modeled by the following formula:

   $P(t) = 1.85(0.985)^t$, where t is the number of years since 1950.

   I can see that this is an exponential decay model because $b < 1$ in the formula $P(t) = a(b)^t$.

   a. According to the model, what was the population of Detroit in 1950?

   $\boldsymbol{P(0) = 1.85(0.985)^0 = 1.85}$

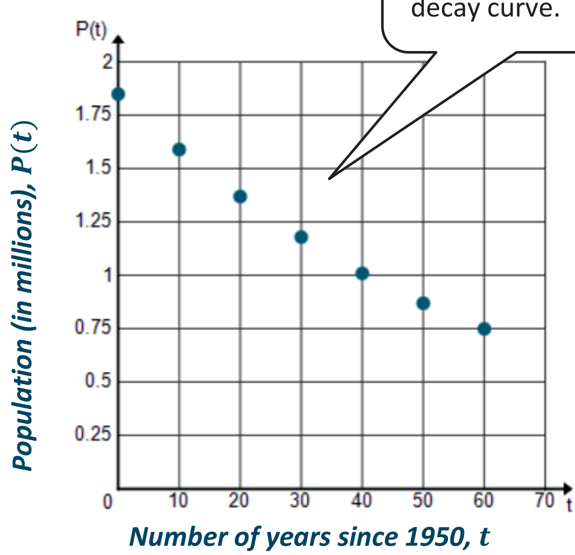
   In 1950, the population of Detroit was approximately $\boldsymbol{1.85}$ million.

   I need to find $P(0)$ since 1950 corresponds to $t = 0$.

   b. Complete the following table, and then graph the points $(t, P(t))$.

   I see that the points form an exponential decay curve.

$\boldsymbol{t}$	$\boldsymbol{P(t)}$
0	$\boldsymbol{1.85}$
10	$\boldsymbol{1.59}$
20	$\boldsymbol{1.37}$
30	$\boldsymbol{1.18}$
40	$\boldsymbol{1.01}$
50	$\boldsymbol{0.87}$
60	$\boldsymbol{0.75}$



c. If this trend continues, estimate the year in which the population of Detroit will be less than 500,000.

I can use trial and error or the table feature on a graphing calculator to find the answer.

$\boldsymbol{P(87) = 1.85(0.985)^{87} \approx 0.497}$

If this trend continues, the population of Detroit will be less than $\mathbf{500,000}$ by the year 2037.

2. A Christmas tree farmer has 6,000 firs on his farm. Each Christmas, he plans to cut down 12% of his trees.

a. Write a formula to model the number of trees on his farm each year.

$\boldsymbol{N(t) = 6000(1 - 0.12)^{t} = 6000(0.88)^{t}}$***, where t represents the number of years.***

If the farmer cuts down 12% of the trees each year, then 88% of the trees are remaining.

b. If he does not plant any new trees, how many trees will he have on his farm in 15 years?

$\boldsymbol{N(15) = 6000(0.88)^{15} \approx 881.843}$

After $\mathbf{15}$ years, the farmer will have approximately $\mathbf{882}$ trees on his farm.